J Math Chem (2011) 49:444–456
DOI 10.1007/s10910-010-9751-x

ORIGINAL PAPER

Mean residence time of Markov processes for particle transport in fludized bed reactors

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Received: 15 October 2009 / Accepted: 1 October 2010 / Published online: 16 October 2010

Abstract A new approach for studying the particle dynamics and RTD (residence time distribution) in processes is to formulate stochastic models. A common question to all models for RTD is whether Danckwerts' law for mean residence time holds. In this paper we revisit a Markov process that has been proposed by Dehling et al. (1999) as a stochastic model for particle transport in fluidized bed reactors. Under the volumetric flow balance conditions, we deduce different boundary conditions at the entrance and the exit of the reactor, and in both discrete model and continuous model we show that processes satisfy Danckwerts' law, stating that the mean residence time of particle transport in fluidized bed reactors equals V/v, where V denotes the volume of the reactor occupied by the fluid and v the volumetric inflow rate.

Keywords Markov process · Stochastic model · Danckwerts' law · Fluidized bed

1 Introduction

In the chemical industry, fluidized bed reactors (see Fig. 1) are widely used and are essential to the production of key commodity and specialty chemicals such as petroleum, pigments, polymers, wastewater treatment and etc. In the reactors, fluidization is the operation by which fine solids are transformed into a fluid-like state through contact with a gas or liquid. However, this method of contacting has a number of unusual characteristics. In spite of their ubiquitous application, understanding of the complex fluidization involved is still limited, and fluidization engineering is concerned with efforts to take advantage of this behavior and put it into good use.

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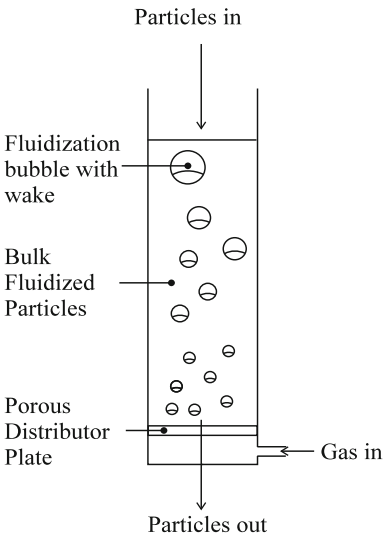

Fig. 1 Processes governing particle transport in a continuous fluidized bed

In reality, we often use mathematical representation to attempt to explain the behavior of some aspect. To serve as a primary approach to the design, optimization, and control of industrial-scale fluidized bed reactors, mathematical models are widely applied as an essential tool to simulate the fluidization phenomena of particle transport in fluidized bed reactors, and result in greatly increased performance and reduced costs associated with fluidized bed implementation and operation.

The mathematical models [6] for particle transport processes can be categorized into two types: one is a deterministic model, and the other is a stochastic model (also known as a probabilistic model). So far, the deterministic model has been used extensively in every many branches of the physical sciences. It performs the same way for a given set of initial conditions, and is based on formulating and solving differential conservation equations. Conversely, in a stochastic model, randomness is present, and variable states are not described by unique values, but rather by probability distributions.

The downside of the deterministic model is it does not fully cover the fact that there is a whole range of possible outcomes and some are more probable and some are less. Moreover, when the process gets more and more complicated, it can be a problem to formulate the model and consume more and more of time for the evaluation. The stochastic model is more realistic as it can account for behavioral factors, and is therefore an attractive alternative way. Using this technique can greatly simplify the formulation of a model, and even make modeling possible for processes so complex that the traditional method is impracticable [5].

For systems such as the transport of particles in fluidized beds, a new approach for studying the particle dynamics in processes is to formulate stochastic models, e.g., [4,11]. For such systems, a stochastic model is in accordance with the inherent random nature of the process, and is intuitively appealing. Since stochastic models focus on all individual particles and models their paths through the fluidized bed reactor, it is also referred to as a "microscopic" approach [3].

Since P. V. Danckwerts original paper [2] the study of residence time distribution (RTD) in fluidized bed reactors with continuous throughflow has become a valuable tool in process technology, and has been investigated in a number of articles. A common question to all models for RTD is whether Danckwerts' law for mean residence time holds. The Danckwerts' law for mean residence time states that the mean residence time of particle transport in fluidized bed reactors equals V/r, where V denotes the volume of the reactor occupied by the fluid and r the volumetric inflow rate.

Answering the question can be a challenge both in numerical and analytical RTD modeling (e.g. [9,12]). In [7], Fan et al. compute residence time distributions for some specific Markov chain models. However, they did not consider the issue of general conditions for the validity of Danckwerts' law, which is the focus of the present paper. In [10], Gottschalk et al. present more general conditions that are sufficient for Danckwerts' law, but only in a discrete model. The continuous model is an interesting open problem, and is also well suited to theoretical research.

In this paper we revisit a Markov process that has been proposed by Dehling et al. [4] as a stochastic model for particle transport in fluidized bed reactors. Under some general conditions of volumetric flow balance, we deduce different boundary conditions at the entrance and the exit of the reactor, and in both discrete model and continuous model we show that processes satisfy the Danckwerts' law.

2 The stochastic model

Fluidized beds are obtained by pressing gas through a distributor plate at the bottom into a reactor containing solid particles. If the gas pressure exceeds a critical level, the bed of particles starts behaving like a fluid, showing e.g. diffusion of particles in the reactor. Particles are continuously added at the top of the reactor and removed at the bottom at the same rate.

In bubbling fluidized beds, both jetsam and flotsam (particles tending to float) are carried in bubble wakes (see Fig. 1), and fluidization bubbles cause a stirring action referred to as "dispersion". The shearing of the bed material due to this stirring action also allows individual jetsam particles to segregate towards the bottom of the bed.

In [4], Dehling et al. have proposed a Markov stochastic model (see Fig. 2) for transport of marked particles in fluidized bed reactors. The attractive features of the models are for instance that they are simple to formulate and easy to compute using a matrix-oriented package such as MATLAB. The stochastic model is based on these particle transport concepts, which are first introduced by Gibilaro et al. in [8]. Researchers in chemical engineering believe that the following three processes account for the transport of individual particles in the reactor.

- Transport upwards in bubble wakes and deposition on top of the bed.
- Transport down in the bulk due to the removal of material low in the bed in bubble wakes ('circulation'), and
- Dispersion due to the disturbance of the bulk material by fluidization bubbles.

The basic stochastic model proposed by Dehling et al. is a Markov chain $(X_n)_{n\geq 0}$ with state space $\{0, 1, \ldots, N\}$ where the states denote the index of the cell in a discretized reactor that is occupied by the particle. The state N corresponds with the exterior of

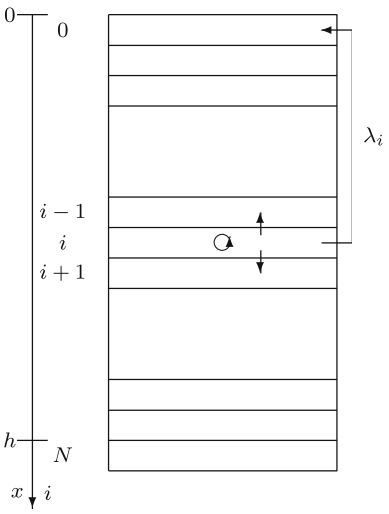

Fig. 2 Transition probabilities in the discrete Markov process model

the reactor that is reached when a particle has left the reactor via the exit at the bottom. The model calculates the probability distribution of the axial position of one particle as a function of time. The four possible transitions are: (1) staying in the same cell, (2) moving to the next cell, (3) moving back to the previous cell, (4) being caught up in a bubble wake and deposited at the top of the bed. The corresponding transition probabilities $p_{i,j} = P(X_{n+1} = j | X_n = i)$ at time n of the Markov chain are specified by

$$\begin{cases} p_{i,i+1} = \beta_i(1-\lambda_i) \\ p_{i,i} = \alpha_i(1-\lambda_i) \\ p_{i,i-1} = \delta_i(1-\lambda_i) \\ p_{i,0} = \lambda_i, \end{cases} \tag{1}$$

where $\beta_i + \delta_i + \alpha_i = 1$. These probabilities hold in the interior of the reactor, i.e. for $1 \leq i \leq N-1$, and reflect the four physical processes governing the particle transport. The probability $p_{i,0}$ specifically reflects the possibility that a particle can be transported upward in the wake of rising fluidization bubbles and deposited at the top of the bed. For $i = 1$, the two probabilities $p_{i,i-1}$ and $p_{i,1}$ are added to yield:

$$p_{1,0} = \lambda_1 + \delta_1(1-\lambda_1)$$

At the entrance of the reactor a reflecting boundary condition was imposed:

$$p_{0,1} = \beta_0(1-\lambda_0)$$
$$p_{0,0} = 1 - \beta_0(1-\lambda_0)$$

The exit boundary in the model is assumed to be absorbing, i.e.

$$p_{N,N} = 1$$

3 Danckwerts' law for mean residence time in the discrete stochastic model

Danckwerts' law states that the mean residence time of particles in a reactor only depends on the volume of the reactor occupied by the fluid and the volumetric inflow/outflow rate, and is given by $\frac{V}{v}$. In this section we will show that the discrete stochastic model proposed by Dehling et al. satisfies Danckwerts' law, provided the return probability and the downward drift are related in such a way that a volume balance holds. In vague terms, we have to require that the net downward flow through the reactor is everywhere equal the inflow/outflow. By net downward flow we mean here the balance of the flow of the bulk of the material with the upward flow in the wake phase.

We denote the volumetric inflow/outflow rate by v, the cross sectional area of the reactor by A and the time and space discretizations by ϵ and Δ, respectively. When the volumetric inflow rate is equal to the volumetric outflow rate, the flow in reactor is in equilibrium. Because of the materials are incompressible, the volumetric balance of flow results in the requirements:

$$\begin{cases} \beta_i = \frac{\delta_{i+1}(1-\lambda_{i+1})+\frac{v\epsilon}{A\Delta}+\sum_{j=i+1}^{N-1}\lambda_j}{1-\lambda_i} & 0 \le i < N-1 \\ \beta_{N-1} = \frac{v\epsilon}{(1-\lambda_{N-1})A\Delta} \end{cases} \tag{2}$$

We define the first exit time T by $\inf\{t \ge 0;\, X_t = N\}$, which denotes the time of arrival at the absorbing boundary. In chemical engineering terminology, T is the residence time of a particle in the reactor.

Theorem 1 *Let $(X_n)_{n\ge 0}$ with state space $\{0, 1, \ldots, N\}$ be a Markov chain with transition probabilities (1), satisfying the volumetric flow balance condition (2). Then*

$$E[T \mid X_0 = 0] = \frac{V}{v}$$

Proof Let $W_i = E[T \mid X_0 = i]$ be the mean time of T, where the subscript i denotes starting position $X_0 = i$. □

Time is discretized into intervals of length ϵ and we assume the process spends one ϵ time to take one step. For a starting position $X_0 = i$ $(0 \le i < N)$, we proceed to carry out a first step analysis, considering separately the four different possibilities for the first step, $X_1 = i-1$, $X_1 = i$, $X_1 = i+1$, $X_1 = 0$, with respective probabilities which can be got from the transition probabilities (1). The absorption always takes at least one step, at least ϵ time. If $X_1 = N$, then no further steps are required. If $X_1 = j < N$ then additional steps are need. By invoking the Markov property, we deduce that this future process proceeding from state j to absorption has an expected time equal to W_j. Weighing this by the transition probability $P_{i,j}$ and then summing all contributions in accordance with the law of total probabilities, we obtain the following joint relation:

$$W_i = \epsilon + P_{i,i-1}W_{i-1} + P_{i,i+1}W_{i+1} + P_{i,i}W_i + P_{i,0}W_0$$

Consider starting positions from cell 0 to cell N respectively, and substitute the probabilities given by the transition probabilities (1), we can obtain equations group as follows:

$$\begin{cases} W_0 &= \epsilon + (1-\lambda_0)(1-\beta_0)W_0 + (1-\lambda_0)\beta_0 W_1 + \lambda_0 W_0 \\ W_i &= \epsilon + (1-\lambda_i)\delta_i W_{i-1} + (1-\lambda_i)\alpha_i W_i \\ & \quad +(1-\lambda_i)\beta_i W_{i+1} + \lambda_i W_0 \qquad 0 < i < N \\ W_N &= 0 \end{cases} \tag{3}$$

Substituting α_i by $1-\delta_i-\beta_i$, and plugging the volumetric flow balance conditions (2) into the above equations group (3), we can obtain $W_N, W_{N-1}, \ldots, W_0$ from bottom to top as follows:

$$\begin{cases} W_N = 0 \\ W_i = \frac{(N-i)\epsilon+\delta_i(1-\lambda_i)W_{i-1}+\sum_{j=i}^{N-1}\lambda_j W_0}{\delta_i A\Delta(1-\lambda_i)+v\epsilon+\sum_{j=i}^{N-1}\lambda_j A\Delta} A\Delta \qquad 0 < i < N \\ W_0 = \frac{NA\Delta}{v} \end{cases} \tag{4}$$

Apparently, V is equal to $NA\Delta$, so $E[T \mid X_0 = 0] = W_0 = \frac{NA\Delta}{v} = \frac{V}{v}$, which establishes the claim of the theorem.

4 Danckwerts' law for mean residence time in the continuous model

Let $\tilde{Q}_\Delta(n, i)$ denote the probability that a particle starting from cell i has left the reactor by n times transition. According to the Chapman-Kolmogorov equations, we have:

$$\tilde{Q}_\Delta(n+1, i) = \sum_{0 \le j \le N} P_{i,j}\tilde{Q}_\Delta(n, j),$$

where $0 \le i \le N$. Because of the absorption at bottom, $\tilde{Q}_\Delta(n, N)$ is 1 for any times transitions. Substituting $P_{i,j}$ by the transition probabilities(1), we can then obtain:

$$\begin{cases} \tilde{Q}_\Delta(n+1, 0) = (1-\lambda_0)(1-\beta_0)\tilde{Q}_\Delta(n, 0) \\ \qquad\qquad + (1-\lambda_0)\beta_0\tilde{Q}_\Delta(n, 1) + \lambda_0\tilde{Q}_\Delta(n, 0) \\ \tilde{Q}_\Delta(n+1, i) = (1-\lambda_i)\delta_i\tilde{Q}_\Delta(n, i-1) + (1-\lambda_i)\alpha_i\tilde{Q}_\Delta(n, i) \\ \qquad\qquad + (1-\lambda_i)\beta_i\tilde{Q}_\Delta(n, i+1) + \lambda_i\tilde{Q}_\Delta(n, 0) \qquad 0 < i < N-1 \\ \tilde{Q}_\Delta(n+1, i) = (1-\lambda_i)\delta_i\tilde{Q}_\Delta(n, i-1) + (1-\lambda_i)\alpha_i\tilde{Q}_\Delta(n, i) \\ \qquad\qquad + (1-\lambda_i)\beta_i + \lambda_i\tilde{Q}_\Delta(n, 0) \qquad i = N-1 \end{cases} \tag{5}$$

Lemma 1 *Let $Q_\Delta(t,x)$ denote the cumulative probability that a particle at height x has left the reactor by time t. If the joint distribution function $Q_\Delta(t,x)$ is same at any height of the same cell during the same time step, then:*

$$Q_\Delta(t,x) = \tilde{Q}_\Delta([t/\epsilon],[x/\Delta]). \tag{6}$$

Proof By the definition of $\tilde{Q}_\Delta(n,i)$, we get

$$\tilde{Q}_\Delta([t/\epsilon],[x/\Delta]) = E(E(Q_\Delta(t',x'|[x'/\Delta]=[x/\Delta]))|[t'/\epsilon]=[t/\epsilon])$$

Since the joint distribution function $Q_\Delta(t,x)$ is same at any height in the same cell at time t', we have

$$E(E(Q_\Delta(t',x'|[x'/\Delta]=[x/\Delta]))|[t'/\epsilon]=[t/\epsilon]) = E(Q_\Delta(t',x|[t'/\epsilon]=[t/\epsilon]))$$

the joint distribution function $Q_\Delta(t,x)$ is same at height x during the same time step, we finally obtain:

$$E(Q_\Delta(t',x|[t'/\epsilon]=[t/\epsilon])) = Q_\Delta(t,x)$$

which gives a proof. □

Applying Lemma 1 to (5), we have:

$$\begin{aligned} Q_\Delta(t+\epsilon,i\Delta) &= (1-\lambda_i)(1-\beta_i)Q_\Delta(t,i\Delta) + (1-\lambda_i)\beta_i Q_\Delta(t,(i+1)\Delta) \\ &\quad +\lambda_i Q_\Delta(t,0) \qquad\qquad i=0 \end{aligned} \tag{7}$$

$$\begin{aligned} Q_\Delta(t+\epsilon,i\Delta) &= (1-\lambda_i)\delta_i Q_\Delta(t,(i-1)\Delta) + (1-\lambda_i)\alpha_i Q_\Delta(t,i\Delta) \\ &\quad +(1-\lambda_i)\beta_i Q_\Delta(t,(i+1)\Delta) + \lambda_i Q_\Delta(t,0) \quad 0<i<N-1 \end{aligned} \tag{8}$$

$$\begin{aligned} Q_\Delta(t+\epsilon,i\Delta) &= (1-\lambda_i)\delta_i Q_\Delta(t,(i-1)\Delta) + (1-\lambda_i)\alpha_i Q_\Delta(t,i\Delta) \\ &\quad +(1-\lambda_i)\beta_i + \lambda_i Q_\Delta(t,0) \qquad\qquad i=N-1 \end{aligned} \tag{9}$$

As $\epsilon\to 0$, $\Delta\to 0$, we expect $Q_\Delta(t,x)$ to converge to the the cumulative probability $Q(t,x)$ of the limit process X_t. From this discrete Markov chain one can obtain a continuous model as a diffusion limit by letting the cell width Δ and the time steps ϵ converge to zero. Let $D(x)$, $v(x)$ and $\lambda(x)$ denote the diffusion coefficient, the drift and the return rate, respectively. According to the literature of Dehling et al. [4], we choose a possible choice of parameters δ_i, α_i, λ_i as following:

$$\begin{cases} \delta_i = \frac{\epsilon}{2\Delta^2}D(i\Delta) - \frac{\epsilon}{2\Delta}v(i\Delta) \\ \alpha_i = 1-\beta_i-\delta_i \\ \lambda_i = \epsilon\lambda(i\Delta) \end{cases} \tag{10}$$

Plugging the volumetric flow balance conditions (2) and the possible choice of parameters (10) into (8), taking $i\Delta$ by x, after performing some elementary transformations,

one can have the following partial differential equation inside the reactor for the limit:

$$
\begin{aligned}
Q_\Delta(t+\epsilon,x) = &\left(1-\frac{\sum_{y=x+\Delta}^{h-\Delta}\epsilon\lambda(y)}{1-\epsilon\lambda(x)}\right)(1-\epsilon\lambda(x))Q_\Delta(t,x)\\
&+\left(\sum_{y=x+\Delta}^{h-\Delta}\epsilon\lambda(y)\right)Q_\Delta(t,x+\Delta)+\epsilon\lambda(x)Q_\Delta(t,0)\\
&+\Bigg(\Bigg(-\frac{-\frac{1}{2}v(x+\Delta)\epsilon(1-\epsilon\lambda(x+\Delta))+\frac{v\epsilon}{A}}{1-\epsilon\lambda(x)}\\
&+\frac{1}{2}v(x)\epsilon\Bigg)(1-\epsilon\lambda(x))Q_\Delta(t,x)-\frac{1}{2}v(x)\epsilon(1-\epsilon\lambda(x))Q_\Delta(t,x-\Delta)\\
&+\left(-\frac{1}{2}v(x+\Delta)\epsilon(1-\epsilon\lambda(x+\Delta))+\frac{v\epsilon}{A}\right)Q_\Delta(t,x+\Delta)\Bigg)\Big/\,\Delta\\
&+\Bigg(\frac{1}{2}D(x)\epsilon(1-\epsilon\lambda(x))Q_\Delta(t,x-\Delta)\\
&+\frac{1}{2}D(x+\Delta)\epsilon(1-\epsilon\lambda(x+\Delta))Q_\Delta(t,x+\Delta)+\Bigg(-\frac{1}{2}D(x)\epsilon\\
&-\frac{1}{2}\frac{D(x+\Delta)\epsilon(1-\epsilon\lambda(x+\Delta))}{1-\epsilon\lambda(x)}\Bigg)(1-\epsilon\lambda(x))Q_\Delta(t,x)\Bigg)\Big/\,\Delta^2
\end{aligned}
$$

where h is the height of the reactor, and it equals $N\Delta$. Subtracting $Q_\Delta(t,x)$ from both side and dividing by ϵ, as $\epsilon\to 0$, we get

$$
\begin{aligned}
\frac{\partial Q(t,x)}{\partial t} = &\left(\sum_{y=x+\Delta}^{h-\Delta}\lambda(y)\right)Q(t,x+\Delta)-Q(t,x)\lambda(x)+\lambda(x)Q(t,0)\\
&-Q(t,x)\left(\sum_{y=x+\Delta}^{h-\Delta}\lambda(y)\right)+\frac{1}{2}\left(Q(t,x)v(x)A-2Q(t,x)v\right.\\
&+Q(t,x)v(x+\Delta)A+2Q(t,x+\Delta)v-Q(t,x+\Delta)v(x+\Delta)A\\
&\left.-v(x)Q(t,x-\Delta)A\right)\Big/(A\Delta)+\frac{1}{2}(-Q(t,x)D(x+\Delta)\\
&+Q(t,x+\Delta)D(x+\Delta)-Q(t,x)D(x)+Q(t,x-\Delta)D(x))\Big/\Delta^2
\end{aligned}
$$

As $\Delta\to 0$, we have

$$
\begin{aligned}
\frac{\partial Q(t,x)}{\partial t} = &(Q(t,0)-Q(t,x))\lambda(x)+\frac{1}{2}\frac{\partial}{\partial x}\left(D(x)\frac{\partial Q(t,x)}{\partial x}\right)\\
&+\frac{\partial Q(t,x)}{\partial x}\left(\frac{v}{A}+\int_x^h\lambda(y)dy\right)
\end{aligned}
\tag{11}
$$

Plugging the volumetric flow balance conditions (2) and the possible choice of parameters (10) into (7), we obtain the following boundary condition at the entrance to the reactor in the limit:

$$
\begin{aligned}
&Q_\Delta(t+\epsilon,0)\\
&=\left(1-\frac{\sum_{y=\Delta}^{h-\Delta}\epsilon\lambda(y)}{1-\epsilon\lambda(0)}\right)(1-\epsilon\lambda(0))Q_\Delta(t,0)+\left(\sum_{y=\Delta}^{h-\Delta}\epsilon\lambda(y)\right)Q_\Delta(t,\Delta)\\
&\quad+\epsilon\lambda(0)Q_\Delta(t,0)+\left(-\frac{1}{2}v(\Delta)\lambda(\Delta)Q_\Delta(t,0)+\frac{1}{2}v(\Delta)\lambda(\Delta)Q_\Delta(t,\Delta)\right)\epsilon^2\\
&\quad+\frac{\left(\frac{1}{2}v(\Delta)A-v\right)Q_\Delta(t,0)+\left(-\frac{1}{2}v(\Delta)A+v\right)Q_\Delta(t,\Delta)}{A\Delta}\epsilon\\
&\quad+\frac{\left(-\frac{1}{2}D(\Delta)\lambda(\Delta)Q_\Delta(t,\Delta)+\frac{1}{2}D(\Delta)\lambda(\Delta)Q_\Delta(t,0)\right)\epsilon^2+(\frac{1}{2}D(\Delta)Q_\Delta(t,\Delta)-\frac{1}{2}D(\Delta)Q_\Delta(t,0))\epsilon}{\Delta^2}
\end{aligned}
$$

Subtracting $Q_\Delta(t,0)$ from both side and multiplying by Δ/ϵ, as $\epsilon\to 0$, we get

$$
\begin{aligned}
\Delta\frac{\partial Q(t,0)}{\partial t}&=(-Q(t,0)\left(\sum_{y=\Delta}^{h-\Delta}\lambda(y)\right)+\left(\sum_{y=\Delta}^{h-\Delta}\lambda(y)\right)Q(t,\Delta))\Delta\\
&\quad-\frac{1}{2}\frac{Q(t,\Delta)v(\Delta)A-Q(t,0)v(\Delta)A-2Q(t,\Delta)v+2Q(t,0)v}{A}\\
&\quad-\frac{1}{2}\frac{-Q(t,\Delta)D(\Delta)+Q(t,0)D(\Delta)}{\Delta}
\end{aligned}
$$

As $\Delta\to 0$, we have

$$
\frac{1}{2}\frac{\partial Q(t,x)}{\partial x}D(x)|_{x=0}=0 \tag{12}
$$

Plugging the volumetric flow balance conditions (2) and the possible choice of parameters (10) into (9), taking $i\Delta$ by x, we obtain the following boundary condition at the exit to the reactor:

$$
\begin{aligned}
&Q_\Delta(t+\epsilon,h-\Delta)\\
&=(-\lambda(h-\Delta)Q_\Delta(t,h-\Delta)+\lambda(h-\Delta)Q_\Delta(t,0))\epsilon+Q(t,h-\Delta)\\
&\quad+\left(\left(\frac{1}{2}v(h-\Delta)\epsilon-\frac{v\epsilon}{(1-\epsilon\lambda(h-\Delta))A}\right)(1-\epsilon\lambda(h-\Delta))Q_\Delta(t,h-\Delta)\right.\\
&\quad\left.-\frac{1}{2}v(h-\Delta)\epsilon(1-\epsilon\lambda(h-\Delta))Q_\Delta(t,h-2\Delta)+\frac{v\epsilon}{A}\right)\Big/\Delta\\
&\quad+\left(\left(-\frac{1}{2}D(h-\Delta)\lambda(h-\Delta)Q_\Delta(t,h-2\Delta)\right.\right.
\end{aligned}
$$

$$+\frac{1}{2}D(h-\Delta)\lambda(h-\Delta)Q_\Delta(t,h-\Delta)\Big)\epsilon^2\left(\frac{1}{2}D(h-\Delta)Q_\Delta(t,h-2\Delta)\right.$$
$$\left.-\frac{1}{2}D(h-\Delta)Q_\Delta(t,h-\Delta)\Big)\epsilon\right)\Big/\Delta^2$$

Subtracting $Q_\Delta(t,h-\Delta)$ from both side and multiplying by Δ/ϵ, as $\epsilon\to 0$, we get

$$\begin{aligned}&\Delta\frac{\partial Q(t,h-\Delta)}{\partial t}\\&=(Q(t,0)\lambda(h-\Delta)-Q(t,h-\Delta)\lambda(h-\Delta))\Delta\\&\quad-\frac{1}{2}\frac{-Q(t,h-\Delta)v(h-\Delta)A-2v+2Q(t,h-\Delta)v+Q(t,h-2\Delta)v(h-\Delta)A}{A}\\&\quad-\frac{1}{2}\frac{-Q(t,h-2\Delta)D(h-\Delta)+Q(t,h-\Delta)D(h-\Delta)}{\Delta}\end{aligned}$$

Let $\Delta\to 0$, we obtain

$$\frac{v-vQ(t,h)}{A}-\frac{1}{2}\frac{\partial Q(t,x)}{\partial x}D(x)|_{x=h}=0 \tag{13}$$

Theorem 2 *The mean residence time in the continuous model satisfy Danckwerts' law, that is*

$$\int_{t=0}^{\infty} t\,dF(t)=\frac{V}{v},$$

where $F(t)$ is the residence time distribution function which gives the probability that a particle entering the reactor at time $t=0$ has left it by time t.

Proof Integrating partial differential Eq. 11 with regard to x from 0 to h, we have

$$\int_0^h\frac{\partial Q(t,x)}{\partial t}dx=\int_0^h(Q(t,0)\lambda(x)-Q(t,x)\lambda(x))dx+\frac{1}{2}D(x)\frac{\partial Q(t,x)}{\partial x}|_{x=0}^h$$
$$+\frac{v}{A}Q(t,x)|_{x=0}^h+\int_0^h\left(\frac{\partial Q(t,x)}{\partial x}\int_x^h\lambda(y)dy\right)dx$$

Using the boundary conditions (12), (13), we obtain

$$\int_0^h\frac{\partial Q(t,x)}{\partial t}dx=\int_0^h(Q(t,0)\lambda(x)-Q(t,x)\lambda(x))dx+\frac{v}{A}-\frac{v}{A}Q(t,0)$$

$$+\int_0^h \left(\frac{\partial Q(t,x)}{\partial x} \int_x^h \lambda(y)dy \right) dx \tag{14}$$

Applying the formula of the integration by parts to the right-most item, we get

$$\begin{aligned}\int_0^h \left(\frac{\partial Q(t,x)}{\partial x} \int_x^h \lambda(y)dy \right) dx &= \int_0^h \left(\int_x^h \lambda(y)dy \right) dQ(t,x) \\ &= Q(t,x) \int_x^h \lambda(y)dy|_{x=0}^h + \int_0^h Q(t,x)\lambda(x)dx \\ &= \int_0^h (Q(t,x)\lambda(x) - Q(t,0)\lambda(x))dx \end{aligned} \tag{15}$$

Plugging (15) into (14), we have

$$\int_0^h \frac{\partial Q(t,x)}{\partial t} dx = \frac{v}{A} - \frac{v}{A} Q(t,0)$$

Integrating t from 0 to ∞ on both sides, this yields

$$\int_0^h (Q(t,x)|_{t=0}^{\infty})dx = \frac{v}{A} \int_0^{\infty} (1 - Q(t,0))dt$$

By time 0, no material can get out of the reactor without time, $Q(0,x)$ is thus zero. All materials starting at height x must eventually get out of the reactor, so $Q(\infty,x)$ is 1, and the left side of the above equation is h. At time 0, the tracer is added into the reactor instantaneously, the value of $Q(t,0)$ is the fraction of tracer which have got out of the reactor by time t, which is the same as the residence time distribution function $F(t)$. We then have

$$\int_0^{\infty} (1 - F(t))dt = \frac{Ah}{v} = \frac{V}{v}$$

Some time later, all those tracers which enter the reactor at time 0 must also generally left the reactor at last, that is to say $F(\infty) = 1$. Applying the formula of the integration by parts, we finally get:

$$\begin{aligned}\frac{V}{v} &= \int_{t=0}^{\infty} (1 - F(t)dt \\ &= t(1 - F(t))|_{t=0}^{\infty} + \int_{t=0}^{\infty} t dF(t) \\ &= \int_{t=0}^{\infty} t dF(t)\end{aligned}$$

the mean residence time of the materials entering the system, which satisfies the Danckwerts' law. □

5 Conclusions

The stochastic model offers great advantages over deterministic models for systems such as the transport of particles in fluidized beds. It is easier to formulate, and simpler and faster to evaluate. Moreover, the stochastic model has some good statistical properties and it is convenience for guiding the nonlinear time series data to the model.

An important part of the modeling process is the evaluation of an acquired model. In this paper, we revisit a Markov process that has been proposed by Dehling et al. [4] as a stochastic model for particle transport in fluidized bed reactors, and prove that in both discrete model and continuous model processes satisfy Danckwerts' law. This shows from another point of view that the stochastic model is suitable for modeling the fluidization phenomena of particle transport in fluidized bed reactors.

It is not very hard to construct the proof presented in this paper by hand. However, to do the algebraic computations, it is really boring and prone to error. Computer algebra systems are good at doing algebraic computations. Among them, Maple [1] is the leading all purpose mathematics software tool. To ease our algebraic computations presented in this paper, we have chosen Maple for mechanical support.

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